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New q-Derivative and q-Logarithm

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In this paper a new q-derivative is proposed and its properties are discussed. We define q-addition and study its axiomatic properties. The q-exponents and q-logarithmic function are introduced and their algebraic structure discussed.

1. INTRODUCTION

The quantum Yang–Baxter equation plays an important role in diverse problems in theoretical physics. These involve exactly soluble models in statistical mechanics (Baxter, 1982) and quantum integrable model field theory (Faddeev, 1981; Bogoliubov *et al.*, 1985; Bullough *et al.*, 1988; Sklynin, 1982; Kulish and Reshetikhin, 1983; de Vega *et al.*, 1984). Just as the Jacobi identity endows a Lie algebra with an associativity condition, the quantum Yang–Baxter equation plays a similar role for a new type of algebraic structure that is a generalization of a Lie algebra.

This structure is sometimes described as a q-deformation of a Lie algebra. From a mathematical point of view, it is a noncommutative Hopf algebra, but in the context of quantum integrable models, it is called a quantum group (Doebner and Henning, 1990; Bullough *et al.*, 1990). The structure and representation of quantum groups have been developed extensively by Jimbo (1985), Drinfel'd (1986), and Faddeev (1984). The deformed analog of the harmonic oscillator has already been studied (Kuryshkin, 1980). The representation theory of the q-deformation of the Lie algebra $su(2) \rightarrow su_q(2)$ has been intensively investigated (Bidenharn, 1989; Macfarlane, 1989; Kulish and Damaskinsky, 1990; Kulish, 1991; Gerdjikov *et al.*, 1984; Arik and Coon, 1976). These papers are concerned

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with the q-deformed boson satisfying the deformed Heisenberg–Weyl algebra, which we call q-bosons.

The q-differentiation and q-integration were first introduced by Jackson (1908, 1910, 1951). He defined the q-number $[n]$ as

$$[n] = \frac{1 - q^n}{1 - q}$$

and tried to find a derivative which is defined so that the derivative of x^n with respect to x is given by

$$[n]x^{n-1}$$

He called this kind of derivative a q-derivative, which is defined as

$$\frac{df(x)}{d_q x} = \frac{f(x) - f(qx)}{x(1 - q)}$$

It is easy to show that the q-derivative of x^n with respect to x is $[n]x^{n-1}$ by using this kind of derivative. Of course, the q-derivative reduces to an ordinary derivative when q goes to 1. Recently, the q-derivative has been used in order to construct the q-analog of Bargmann space and coherent states (Gray and Nelson, 1990; Bracken *et al.*, 1991).

In this paper we define q-addition and discuss its properties. We use this to propose a new q-derivative and to construct a q-logarithmic function and q-exponents and to investigate their properties.

2. q-ADDITION

In this section we define q-addition and discuss its properties. Let us define q-addition by

$$(a \oplus_q b)^n = \sum_{k=0}^{n} {}_nC_k^q a^k b^{n-k} \qquad (a \neq b) \tag{1}$$

$$(a \oplus_q a)^n = (a + a)^n = 2^n a^n \tag{2}$$

where

$${}_nC_k^q = \frac{[n]!}{[k]![n-k]!}$$

$$[n] = \frac{1 - q^n}{1 - q}$$

From now on we assume that a is different from b. If we write the first few terms corresponding to $n = 0, 1, 2, \ldots$, then we have

$$(a \oplus_q b)^0 = 1$$

$$(a \oplus_q b)^1 = a + b$$

$$(a \oplus_q b)^2 = a^2 + [2]ab + b^2$$

$$(a \oplus_q b)^3 = a^3 + [3]a^2b + [3]ab^2 + b^3$$

It is important to notice that $a \oplus_q b$ does not mean the same thing as $(a \oplus_q b)^1$. From the above definition we have the following properties:

$$\begin{aligned} a \oplus_q b &= b \oplus_q a \\ a \oplus_q 0 &= 0 \oplus_q a = a \\ ka \oplus_q kb &= k(a \oplus_q b) \end{aligned} \tag{3}$$

The first property of (3) is commutativity property of q-addition, the second property means that there exists an identity element of q-addition, and the third property is a distributive property. The q-subtraction is defined as

$$\begin{aligned} &q \ominus_q b = a \oplus_q (-b), \qquad a \neq b \\ &a \ominus_q a = a - a = 0 \\ &a \ominus_q b = -(b \ominus_q a) \end{aligned} \tag{4}$$

where the last property of (4) is proved as follows:

$$\begin{aligned} a \ominus_q b &= a \oplus_q (-b) \\ &= (-1)^2 a \oplus_q (-b) \\ &= -(a \oplus_q b) \\ &= -(b \oplus_q (-a)) \\ &= -(b \ominus_q a) \end{aligned}$$

Now we generalize the above discussion to the n different variables $x_1, x_2, \ldots, x_n$. Then the multinomial expansion is written as

$$(x_1 \oplus_q x_2 \oplus_q \cdots \oplus_q x_n)^N = ((x_1 \oplus_q x_2 \oplus_q \cdots \oplus_q x_{n-1}) \oplus_q x_n)^N \tag{5}$$

Using the definition of q-binomial expansion, we can rewrite equation

(5) as

$$(x_1 \oplus_q x_2 \oplus_q \cdots \oplus_q x_n)^N$$
$$= \sum_{k_2=0}^{N} \sum_{k_3=0}^{N-k_2} \cdots \sum_{k_n=0}^{N-\sum_{i=2}^{n-1} k_1} \frac{[N]!}{[k_2]! \cdots [k_n]![N-\sum_{i=2}^{n} k_i]!}$$
$$\times x_1^{k_1} \cdots x_{n-1}^{k_{n-1}} x_n^{N-\sum_{i=1}^{n-1} k_i}$$
$$= \sum_{k_1=0}^{N} \sum_{k_2=0}^{N} \cdots \sum_{k_n=0, k_1+\cdots+k_n=N}^{N} \frac{[N]!}{\prod_{i=1}^{N} [k_i]!} x_1^{k_1} \cdots x_n^{k_n} \qquad (6)$$

For example, let us compute such a quantity as

$$A = (1 \oplus_q 2 \oplus_q 3)^2$$

By definition, A means

$$A = ((1 \oplus_q 2) \oplus_q 3)^2$$

It is worth noting that

$$A \neq (3 \oplus_q 3)^2$$

From the definition (1), we have

$$\begin{aligned} A &= (1 \oplus_q 2)^2 + [2](1 \oplus_q 2)^1 \cdot 3 + 3^2 \\ &= 1^2 + 2^2 + 3^2 + [2](1 \cdot 2 + 1 \cdot 3 + 2 \cdot 3) \\ &= 25 + 11q \end{aligned}$$

On the other hand, we define the q-exponential as

$$e_q(x) = \sum_{n=0}^{\infty} \frac{1}{[n]!} x^n \qquad (7)$$

Then we have

$$e_q(x_1) e_q(x_2) = e_q(x_1 \oplus_q x_2) \qquad (8)$$

where the right-hand side of the above equation is defined as

$$e_q(x_1 \oplus_q x_2) = \sum_{n=0}^{\infty} \frac{1}{[n]!} (x_1 \oplus_q x_2)^n \qquad (9)$$

This can be easily generalized into the form

$$e_q(x_1) e_q(x_2) \cdots e_q(x_N) = e_q(x_1 \oplus_q x_2 \cdots \oplus_q x_N) \qquad (10)$$

where the right-hand side of equation (10) is given by

$$e_q(x_1 \oplus_q x_2 \oplus_q \cdots \oplus_q x_N) = \sum_{n=0}^{\infty} \frac{1}{[n]!}(x_1 \oplus_q x_2 \oplus_q \cdots \oplus_q x_N)^n \quad (11)$$

where $x_1, x_2, \ldots, x_N$ are n different variables.

3. NEW q-DERIVATIVE

In this section we propose a new q-derivative and investigate some of its properties. Let us define the q-derivative of a function $f(x)$ with respect to x as

$$D_x f(x) = \lim_{\delta x \to 0} \frac{f(x \oplus_q \delta x) - f(x)}{\delta x} \quad (12)$$

According to the new definition of q-derivative (12), we have

$$\begin{aligned} D_x x^n &= \lim_{\delta x \to 0} \frac{(x \oplus_q \delta x)^n - x^n}{\delta x} \\ &= \lim_{\delta x \to 0} \frac{(x^n + [n]x^{n-1}\delta x + O(\delta x^2)) - x^n}{\delta x} \\ &= [n]x^{n-1} \end{aligned} \quad (13)$$

Similarly we have

$$D_x(x \oplus_q a)^n = [n](x \oplus_q a)^{n-1} \quad (14)$$

Let us consider the q-derivative of a function $f(x) = 1/x^k$ with respect to x,

$$\begin{aligned} D_x\left(\frac{1}{x^k}\right) &= \lim_{\delta x \to 0} \frac{1/(x \oplus_q \delta x)^k - 1/x^k}{\delta x} \\ &= -\lim_{\delta x \to 0} \frac{(x \oplus_q \delta x)^k - x^k}{\delta x (x \oplus_q \delta x)^k x^k} \\ &= -\frac{[k]}{x^{k+1}} \end{aligned} \quad (15)$$

which is different from the earlier result (Arik and Coon, 1976). Then the q-derivative is defined as

$$\frac{df(x)}{d_q x} = \frac{f(x) - f(qx)}{(1-q)x}$$

Therefore the q-derivative of $f(x) = 1/x^k$ is then given by

$$\frac{d}{d_q x}\left(\frac{1}{x^k}\right) = \frac{[-k]}{x^{k+1}} = -\frac{q^{-k}[k]}{x^{k+1}}$$

which leads to results different from equation (15). Similarly, we obtain

$$D_x \frac{1}{(x \oplus_q a)^k} = -\frac{[k]}{(x \oplus_q a)^{k+1}} \tag{16}$$

The q-differentiation of the q-exponential function is given by

$$\begin{aligned} D_x e_q(x) &= \lim_{\delta x \to 0} \frac{e_q(x \oplus_q \delta x) - e_q(x)}{\delta x} \\ &= \lim_{\delta x \to 0} \frac{e_q(x) e_q(\delta x) - e_q(x)}{\delta x} \\ &= e_q(x) \lim_{\delta x \to 0} \frac{(1 + \delta x + O(\delta x^2)) - 1}{\delta x} \\ &= e_q(x) \end{aligned} \tag{17}$$

Similarly we obtain

$$D_x e_q\left(\frac{a}{x}\right) = -\frac{a}{x^2} e_q\left(\frac{a}{x}\right) \tag{18}$$

From the new definition of q-differentiation we have the following Leibniz rule:

$$D_x(f(x)g(x)) = D_x f(x) \cdot g(x) + f(qx) D_x g(x) \tag{19}$$

where we should note that

$$f(x \oplus_q \delta x) g(x \oplus_q \delta x) \neq [f(x)g(x)]_{x \to x \oplus_q \delta x}$$

and the proof is given in Appendix A.

Now we discuss the q-Taylor expansion. Let us assume that an arbitrary regular function is written as

$$f(x) = \sum_{n=0}^{\infty} c_n (x \ominus_q a)^n \tag{20}$$

q-Differentiating equation (20) with respect to x k times leads to

$$D_x^k f(x) = \sum_{n=k}^{\infty} c_n \frac{[n]!}{[n-k]!} (x \ominus_q a)^{n-k} \tag{21}$$

Inserting $x = a$ in both sides of equation (21) we get

$$[D_x^k f(x)]_{x=a} = c_k [k]!$$

where we used the relation

$$(a \ominus_q a)^n = 0$$

Therefore the coefficient c_k is given by

$$c_k = \frac{1}{[k]!}(D_x^k f(x))_{x=a} \tag{22}$$

4. q-LOGARITHM

In this section we define the q-logarithmic function as the inverse function of the q-exponential function. Now consider the relation

$$x = e_q(y) \tag{23}$$

In this case we write y in terms of x as

$$y = L_q(x) \tag{24}$$

that is,

$$L_q(e_q(x)) = e_q(L_q(x)) = x \tag{25}$$

From now on we will refer to L_q as the q-logarithm. Since $e_q(0) = 1$, we obtain

$$L_q(1) = 0$$

Then the q-logarithm satisfies the following properties:

$$L_q(ab) = L_q(a) \oplus_q L_q(b) \tag{26}$$

$$L_q\left(\frac{a}{b}\right) = L_q(a \ominus_q L_q(b) \tag{27}$$

$$L_q(a^r) = rL_q(a) \tag{28}$$

The proof of equation (26) is as follows. Using the relation

$$e_q(x)e_q(y) = e_q(x \oplus_q y)$$

and the definition of the q-logarithm, we have

$$\begin{aligned} &e_q(L_q(e_q(x)))e_q(L_q(e_q(y))) \\ &= e_q(L_q(e_q(x)) \oplus_q L_q(e_q(y))) \\ &= e_q(L_q(e_q(x)e_q(y))) \end{aligned} \tag{29}$$

Hence we get

$$L_q(e_q(x)) \oplus_q L_q(e_q(y)) = L_q(e_q(x)e_q(y)) \tag{30}$$

Substituting $e_q(x) = a$ and $e_q(y) = b$, we obtain

$$L_q(ab) = L_q(a) \oplus_q L_q(b) \tag{31}$$

The relation (27) is similarly proved. Now we must prove the relation (28).

The left hand side of equation (28) is written as

$$\begin{aligned} L_q(a^r) &= (L_q(a \cdot a \cdots a) \\ &= L_q(a) \oplus_q L_q(a) \oplus_q \cdots \oplus_q L_q(a) \\ &= rL_q(a) \end{aligned} \tag{32}$$

where we used the relation (2).

When $L_q(x) = 1$ holds, let us write x as

$$x = e_q$$

From the definition, we obtain

$$L_q(e_q) = 1 \rightarrow e_q = e_q(1) = \sum_{n=0}^{\infty} \frac{1}{[n]!}$$

which goes back to an ordinary Euler number e when the deformation parameter q goes to 1.

5. q-EXPONENTS

In this section we introduce the q-exponents in terms of the q-logarithmic function as

$$a(r) = e_q(rL_q(a)) \tag{33}$$

When q goes to 1, equation (33) reduces to

$$a(r) \rightarrow e^{r \ln a} = a^r$$

The q-exponents $a(r)$ satisfy the following properties:

$$\begin{aligned} &\text{I.} \quad a(x)a(y) = a(x \oplus_q y) \\ &\text{II.} \quad \frac{a(x)}{a(y)} = a(x \ominus_q y) \\ &\text{III.} \quad (ab)(x) = a(x)b(x) \\ &\text{IV.} \quad \left(\frac{a}{b}\right)(x) = \frac{a(x)}{b(x)} \end{aligned} \tag{34}$$

The proof of property I is as follows:

$$\begin{aligned} a(x)a(y) &= e_q(xL_q(a))e_q(yL_q(a)) \\ &= e_q(xL_q(a) \oplus_q yL_q(a)) \\ &= e_q((x \oplus_q y)L_q(a)) \\ &= a(x \oplus_q y) \end{aligned}$$

The proof of property II is obtained in a similar manner. It is very easy to verify property III:

$$\begin{aligned}(ab)(x) &= e_q(xL_q(ab)) \\ &= e_q(x(L_q(a) \oplus_q L_q(b))) \\ &= e_q(xL_q(a))e_q(xL_q(b)) \\ &= a(x)b(x)\end{aligned}$$

Similarly we can prove property IV, but we omit it for the sake of brevity.

6. CONCLUSION

In this paper we propose a new kind of q-derivative and discuss some of its properties. We define q-addition as different from the ordinary addition and discuss its properties. We use the q-addition to define a q-logarithmic function and q-exponents and investigate their properties.

APPENDIX A. PROOF OF q-LEIBNIZ RULE

In this appendix we derive the property (19). If $f(x)$ and $g(x)$ are written as

$$f(x) = \sum_n a_n x^n$$

$$g(x) = \sum_m b_m x^m$$

then $f(x)g(x)$ is given by

$$f(x)g(x) = \sum_n \sum_m a_n b_m x^{n+m}$$

Therefore its q-derivative is given by

$$\begin{aligned}D_x(f(x)g(x)) &= \lim_{\delta x \to 0} \frac{\sum_n \sum_m a_n b_m (x \oplus_q \delta x)^{n+m} - \sum_n \sum_m a_n b_m x^{n+m}}{\delta x} \\ &= \sum_n \sum_m a_n b_m [n+m] x^{n+m-1} \\ &= \sum_n [n] a_n x^{n-1} \sum_m b_m x^m + \sum_n a_n (qx)^n \sum_m [m] b_m x^{m-1} \qquad \text{(A1)}\end{aligned}$$

where we used the relation

$$[n+m] = [n] + q^n[m]$$

Hence we have proved the relation (19).

APPENDIX B

In the appendix, though not used in this paper, we prove the following restricted q-chain rule for the q-derivative:

$$D_x f(u(x)) = D_u f(u) * D_x u(x) \tag{B1}$$

where

$$u(x) = x^k \oplus_q 1$$

and

$$x^{kl} * [k] = [k]_{q^{l+1}} x^{kl}$$

Proof. Let $f(u)$ be expressed as

$$\begin{aligned} f(u) &= \sum_{n=0}^{\infty} a_n u^n \\ &= \sum_{n=0}^{\infty} a_n (x^k \oplus_q 1)^n \\ &= \sum_{n=0}^{\infty} a_n \sum_{l=0}^{n} {}_nC_l^q x^{kl} \end{aligned} \tag{B2}$$

Then we have

$$D_x f(u) = \sum_{n=0}^{\infty} a_n \sum_{l=1}^{n} {}_nC_l^q [kl] x^{kl-1} \tag{B3}$$

The right-hand side of equation (36) is given by

$$\begin{aligned} \text{RHS} &= \sum_{n=0}^{\infty} a_n [n] u^{n-1} * [k] x^{k-1} \\ &= \sum_{n=0}^{\infty} a_n [n] \sum_{l=0}^{n-1} {}_{n-1}C_l^q x^{kl} * [k] x^{k-1} \\ &= \sum_{n=0}^{\infty} a_n [n] \sum_{l=0}^{n-1} {}_{n-1}C_l^q [k]_{q^{l+1}} x^{k(l+1)-1} \end{aligned}$$

Substituting $l \to l - 1$ leads to

$$= \sum_{n=0}^{\infty} a_n [n] \sum_{l=1}^{n} {}_{n-1}C_{l-1}^{q} [k]_{q^l} x^{kl-1}$$

$$= \sum_{n=0}^{\infty} a_n \sum_{l=1}^{n} \frac{[n][n-1]![k]_{q^l}}{[n-l]![l-1]!} x^{kl-1}$$

$$= \sum_{n=0}^{\infty} a_n \sum_{l=1}^{n} {}_{n}C_{l}^{q} [l][k]_{q^l} x^{kl-1}$$

$$= \sum_{n=0}^{\infty} a_n \sum_{l=1}^{n} {}_{n}C_{l}^{q} [kl] x^{kl-1} = \text{LHS}$$

which completes the proof of equation (B1).

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